Chapter 5

Boundary Value Problem

In this chapter I will consider the so-called boundary value problem (BVP), i.e., a differential equation plus boundary conditions. For example, finding a heteroclinic trajectory connecting two equilibria $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$ is actually a BVP, since I am looking for $\boldsymbol{x}$ such that $\boldsymbol{x}(t) \to \hat{\boldsymbol{x}}_1$ when $t \to \infty$ and $\boldsymbol{x}(t) \to \hat{\boldsymbol{x}}_2$ when $t \to -\infty$. Another example when BVP appear naturally is the study of periodic solutions. Recall that the problem $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$, where $\boldsymbol{f}$ is $\mathcal{C}^{(1)}$ and T-periodic with respect to t, has a T-periodic solution $\boldsymbol{\phi}$ if and only if $\boldsymbol{\phi}(0) = \boldsymbol{\phi}(T)$. Some BVP may have a unique solution, some have no solutions at all, and some may have infinitely many solution.

Exercise 5.1. Find the minimal positive number T such that the boundary value problem

$$x'' - 2x' = 8\sin^2 t, \quad x'(0) = x'(T) = -1,$$

has a solution.

5.1 Motivation I: Wave equation

Arguably one of the most important classes of BVP appears while solving partial differential equations (PDE) by the *separation of the variables* (Fourier method). To motivate the appearance of PDE, I will start with a system of ordinary differential equations of the second order, which was studied by Joseph-Louis Lagrange in his famous "Analytical Mechanics" published first in 1788, where he used, before Fourier, the separation of variables[1] technique.

Consider a linear system of $k + 1$ equal masses m connected by the springs with the same spring constant (rigidity) c. Two masses at the ends of the system are fixed. Let $u_j(t)$ be the displacement from the equilibrium position of the j-th mass at time t. Using the second Newtons law and Hook's law for the springs, I end up with the system

$$m\ddot{u}_j = c(u_{j+1} - u_j) - c(u_j - u_{j-1}) = c(u_{j+1} - 2u_j + u_{j-1}), \quad j = 1, \ldots, k-1,$$

with the initial conditions

$$u_j(0) = \alpha_j, \quad u_j'(0) = \beta_j, \quad j = 1, \ldots, k-1,$$

[1]As far as I am aware this method appeared first in one of Daniel Bernoulli's works on vibration of strings in 1753, together with "Fourier's series." "The Analytic Theory of Heat" by Joseph Fourier was published in 1822.

and the boundary conditions (the ends are fixed)

$$u_0(t) = u_k(t) = 0, \quad t \in \mathbf{R}.$$

This problem can be solved directly, by using the methods from Chapter 3. However, Lagrange, following Bernoulli, makes the following assumption: He assumes that the unknown function u_j can be represented as

$$u_j(t) = T(t)J(j),$$

i.e., as the product of functions, the first of which depends only on t and the second one depends only on the integer argument j. On plugging this expression into the original system and rearranging I find

$$\frac{m\ddot{T}(t)}{cT(t)} = \frac{J(j+1) - 2J(j) + J(j-1)}{J(j)}\,.$$

Since the left-hand side depends only on t whereas the right-hand side depends only on j then they both must be equal to the same constant, which I will denote $-\lambda$. Therefore, I get

$$\ddot{T} + \frac{c\lambda}{m}T = 0,$$

and

$$-(J(j+1) - 2J(j) + J(j-1)) = \lambda J(j).$$

Moreover, from the boundary conditions for the original equation, it follows that

$$J(0) = J(k) = 0.$$

These boundary conditions plus the previous equation is actually an *eigenvalue problem* for a *linear difference operator*, which acts on function, defined on integers. To emphasize this I denote

$$\Delta^2 \colon f(j) \mapsto -(f(j+1) - 2f(j) + f(j-1)),$$

and hence my problem can be rewritten as

$$\Delta^2 J = \lambda J, \quad J(0) = J(k) = 0.$$

My goal is thus to determine those values of the constant λ for which my problem has non-trivial (different from zero) solutions.

The general theory of linear difference equations with constant coefficients is very similar to the theory of linear ODE with constant coefficients. In particular, the general solution to the homogeneous equation is given as a linear combination of two linearly independent solutions. Whereas I used the ansatz e^{qt} for ODE, for the difference equations one should use q^j. So, by plugging $J(j) = q^j$ into my equation and canceling I find

$$-(q^2 - 2q + 1) = \lambda q,$$

or, after rearranging,

$$q^2 - 2\left(1 - \frac{\lambda}{2}\right)q + 1 = 0.$$

Instead of directly solving this quadratic equation, I will start with a qualitative analysis.

If my equation has two real roots $q_1 \neq q_2 \in \mathbf{R}$ then the general solution is $J(j) = C_1 q_1^j + C_2 q_2^j$ and the boundary conditions would imply that

$$C_1 + C_2 = 0,$$
$$C_1 q_1^k + C_2 q_2^k = 0,$$

which has a nontrivial solution $(C_1, C_2) \neq (0, 0)$ if and only if the corresponding determinant is zero, which, by direct calculation is impossible for $q_1 \neq q_2$. The case $q_1 = q_2 \in \mathbf{R}$ is analyzed similarly, only in this case the general solution is $J(j) = C_1 q_1^j + C_2 j q_1^j$.

Exercise 5.2. If the reader is unfamiliar with difference equations it may be helpful to consider an explicit computation. Consider, for instance, the famous problem for Fibonacci numbers $F(j), j = 0, 1, \dots$, which are defined as

$$F(j+1) = F(j) + F(j-1), \quad j = 1, 2, \dots, \quad F(0) = 0,\, F(1) = 1,$$

producing the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, \dots$$

This is an example of a difference equation. Solve it by making an educated guess $F(j) = q^j$, and using the initial conditions to determine constants in the general solution

$$F(j) = A q_1^j + B q_2^j.$$

How many digits does $F(1000)$ have?

Back to the analysis of the system of masses. As a final possibility I assume that $q_1 = \overline{q}_2 \in \mathbf{C}$, i.e., $q_{1,2} = re^{\pm \mathrm{i}\theta}$. But from my quadratic equation by Vieta's theorem I have $q_1 q_2 = 1$ and hence $r = 1$. Now

$$C_1 + C_2 = 0,$$
$$C_1 e^{\mathrm{i}\theta k} + C_2 e^{-\mathrm{i}\theta k} = 0,$$

and the corresponding determinant is, up to a multiplicative constant,

$$\sin \theta k,$$

which is equal to zero only if

$$\theta_l = \frac{\pi l}{k}, \quad l \in \mathbf{Z}.$$

I determined my θ_l, but the original constant is λ, which should be expressed through θ_l. Again from Vieta's theorem I have

$$1 - \frac{\lambda_l}{2} = \frac{q_1 + q_2}{2} = \cos \theta_l,$$

hence

$$\lambda_l = 2(1 - \cos \theta_l) = 4 \sin^2 \frac{\theta_l}{2} = 4 \sin^2 \frac{\pi l}{2k},$$

and hence, to have different constants, I should only take $l = 1, \dots, k-1$ (carefully note that the case $\theta_l = 0$ is excluded by the previous analysis).

Therefore, I found that my difference operator Δ^2 with the trivial boundary conditions has exactly $k-1$ eigenvalues λ_l with $k-1$ corresponding eigenfunctions J_l.

Returning to the equation for the time dependent variable, I hence have

$$\ddot{T}_l+\frac{c\lambda_l}{m}T_l=0 \implies T_l=A_l\sin\omega_l t+B_l\cos\omega_l t=C_l\sin(\omega_l t+\varphi_l),\quad w_l=2\sqrt{\frac{c}{m}}\sin\frac{\pi l}{2k}\,,\quad l=1,\dots,k-1.$$

The full solution to the original system can be determined by noting that any linear combination of $(t,j)\mapsto T_l(t)J_l(j)$ is also a solution, and hence

$$u_j(t)=\sum_{l=1}^{k-1}T_l(t)J_l(j)=\sum_{l=1}^{k-1}C_l\sin(\omega_l t+\varphi_l)J_l(j),\quad j=1,\dots,k-1.$$

These solutions have $2(k-1)$ arbitrary constants C_l,φ_l, $l=1,\dots,k-1$, which are uniquely determined by the initial conditions for $k-1$ moving masses.

In a nutshell, an attempt to solve my original problem by the separation of variables method ended up with an eigenvalue problem for a linear difference operator, or, in a slightly different language, in a boundary value problem for a linear difference equation.

Now I will consider the limit when $k\to\infty$. I will assume that the two fixed masses are located at the points 0 and 1 and distance between any two masses is equal to h. Hence, $k\to\infty$ is equivalent to $h\to 0$. From physics I know that $m=\rho Sh$, where ρ is the density and S is the area of the transverse section; $c=ES/h$, where E is called *Young's modulus*. I also introduce new notation

$$u_j(t)=u(t,x_j),$$

emphasizing that when $h\to 0$, the j-th equilibrium position turns into a point of a *continuous* variable $x\in[0,1]$. Using the introduced notations I have (notation $u''_{tt}=u_{tt}=\partial_{tt}u$ means the second partial derivative of u with respect to variable t)

$$u''_{tt}(t,x_j)=\frac{E}{\rho}\frac{u(t,x_{j+1})-2u(t,x_j)+u(t,x_{j-1})}{h^2}\,.$$

The left hand side approaches u_{tt} as $h\to 0$. The right-hand side approaches a^2u_{xx}, where $a=\sqrt{E/\rho}$.

Exercise 5.3. Show that, assuming that u is sufficiently smooth, that

$$\frac{u(t,x_{j+1})-2u(t,x_j)+u(t,x_{j-1})}{h^2}\to\partial_{xx}u(t,x),\quad \text{as } h\to 0.$$

Hint: consider $u(t,x_{j+1})=u(t,x_j+h),u(t,x_{j-1})=u(t,x_j-h)$ and represent them as Taylor's series around $h=0$, simplify and take the limit.

Therefore, I showed that my system of ODE turns into the so-called *wave equation*

$$u_{tt}=a^2u_{xx},\quad u(t,0)=u(t,1)=0,\quad t>0,$$

for the unknown function u of two variables: time t and position x. I also must have the initial conditions

$$u(0,x)=f(x),\quad u_t(0,x)=g(x),\quad 0\le x\le 1.$$

Now, following Lagrange, I will apply the same trick with the separation of variables to the wave equation assuming that

$$u(t,x) = T(t)X(x).$$

After separating the variables I will find for X (the student should fill in all the details) that

$$X'' + \lambda X = 0, \quad X(0) = X(1) = 0.$$

This problem can be written as the eigenvalues problem

$$LX = \lambda X, \quad X(0) = X(1),$$

for the differential operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. This problem has nontrivial (distinct) solutions if and only if

$$\lambda_l = (\pi l)^2, \quad l = 1, 2, \dots$$

These are the eigenvalues of my differential operator. The corresponding eigenfunctions (up to multiplicative constants) are

$$X_l(x) = \sin \pi l x, \quad l = 1, 2, \dots$$

Exercise 5.4. Consider the BVP

$$X'' + \lambda X = 0, \quad X(0) = X(1) = 0,$$

where $\lambda \in \mathbf{C}$. Prove that this problem has nontrivial (different from zero) solutions only if $\lambda_l = (\pi l)^2$, $l = 1, 2, \dots$ (note that zero is not included).

The equation for T_l hence takes the form

$$T_l'' + (\pi l a)^2 T_l = 0,$$

which has the solution

$$T_l(t) = A_l \sin \pi l a t + B_l \cos \pi l a t.$$

Finally, using the superposition principle, I have that the *infinite* linear combination

$$u(t,x) = \sum_{l=1}^{\infty} T_l(t) X_l(x) = \sum_{l=1}^{\infty} (A_l \sin \pi l a t + B_l \cos \pi l a t) \sin \pi l x$$

formally solves my problem and satisfies the boundary conditions. At this point I do not want to bother about the conditions under which infinite series converges, and hence I write "formally." Instead, assuming the validity of my formula, I use the initial conditions to determine A_l, B_l.

For instance, using the first initial condition I find

$$\sum_{l=1}^{\infty} B_l \sin \pi l x = f(x),$$

which means that B_l are *the Fourier coefficients* (if the student did not see the Fourier series before, they should simply ignore the title and concentrate on actual computations) of f of its Fourier sine series. To find these coefficients I will use the orthogonality of sines:

$$\int_0^1 \sin \pi n x \sin \pi m x \, \mathrm{d}x = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & n = m. \end{cases}$$

Therefore,

$$B_l = 2\int_0^1 f(x)\sin \pi l x \, \mathrm{d}x, \quad l = 1, 2, \ldots$$

Similarly,

$$A_l = \frac{2}{\pi l a}\int_0^1 g(x)\sin \pi l x \, \mathrm{d}x, \quad l = 1, 2, \ldots,$$

and the original problem solved (at least on the formal level). There are a lot of questions here about the convergence, completeness of the system of sines, etc. I will answer all these questions from a general and abstract point of view in the following.

Exercise 5.5. The introduced above boundary conditions are not the only possible choice. Consider separately the following four different cases, where in the final case I assume that $x \in [-\pi, \pi]$ for symmetry:

1. $X'(0) = X'(1) = 0$
2. $X(0) = X'(1) = 0$
3. $X'(0) = X(1) = 0$
4. $X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi)$

Find all the values of λ's (eigenvalues) for which the corresponding BVP has a non-trivial solution, the corresponding solutions (eigenfunctions), and check that the set of all solutions forms an orthogonal family with respect to the standard inner product in $\mathcal{L}^2[0,1]$ for the first three problems and in $\mathcal{L}^2[-\pi,\pi]$ for the last one (recall that two functions f, g are orthogonal in $\mathcal{L}^2[a,b]$ if $\int_a^b fg = 0$; the system (ϕ_n) is orthogonal if every ϕ_n is orthogonal to any ϕ_m for all $n \neq m$).

Exercise 5.6. Solve

$$u_{tt} = u_{xx}, \quad t > 0, \quad x \in (0,1),$$

with the boundary condition

$$u(t,0) = u(t,1) = 0,$$

and initial condition

$$u(0,x) = \sin \pi x + 0.5 \sin 3\pi x, \quad u_t(0,x) = 0.$$

Remark 5.1. In general, if I consider a more complicated system of masses connected by springs, I can arrive to the n-dimensional wave equations. For instance, if my masses are placed at the vertices of three dimensional lattice, and each mass is connected to its 6 neighbors, I can obtain by exactly the same limiting procedure that

$$u_{tt} = a^2 \Delta u,$$

where Δ is the *Laplace operator*, in Cartesian coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,.$$

This is three dimensional wave equation.

5.2 Motivation II: Heat equation

Here I will show how a PDE can be derived *macroscopically*, opposite to the *microscopic* approach, which I used in the previous section. I will assume that the reader is familiar with some basic theorems of multivariable calculus.

Let $\boldsymbol{x} \in \Omega \subset \mathbf{R}^3$, and Ω be a bounded domain with boundary $\partial\Omega$. Let $e(t, \boldsymbol{x})$ be the thermal energy at time t at the point $\boldsymbol{x} \in \Omega$. The law of the *conservation of energy* tells me that the rate of change of the total thermal energy in Ω must be equal to the flux through the boundary plus the amount of energy created inside Ω, mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\Omega} e(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = -\oiint_{\partial\Omega} \boldsymbol{q}(t, \boldsymbol{x}) \cdot \boldsymbol{n} \,\mathrm{d}S + \iint_{\Omega} f^*(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{q} \colon \mathbf{R} \times \Omega \longrightarrow \Omega$ is the flux (note, it is a vector), $\boldsymbol{n}$ is the outward normal to the boundary $\partial\Omega$, $\mathrm{d}S$ is the element of the area of $\partial\Omega$, and f^* is the density of sources of the thermal energy (it can be negative representing dissipation of energy). By Gauss' theorem

$$\oiint_{\partial\Omega} \boldsymbol{q}(t, \boldsymbol{x}) \cdot \boldsymbol{n} \,\mathrm{d}S = \iint_{\Omega} \nabla \cdot \boldsymbol{q}(t, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},$$

and hence

$$\iiint_{\Omega} \Big(e_t(t, \boldsymbol{x}) + \nabla \cdot \boldsymbol{q}(t, \boldsymbol{x}) - f^*(t, \boldsymbol{x}) \Big) \,\mathrm{d}\boldsymbol{x} = 0,$$

therefore,

$$e_t(t, \boldsymbol{x}) + \nabla \cdot \boldsymbol{q}(t, \boldsymbol{x}) - f^*(t, \boldsymbol{x}) = 0.$$

I know that $e(t, \boldsymbol{x}) = c(\boldsymbol{x})\rho(\boldsymbol{x})u(t, \boldsymbol{x})$, where c is the thermal capacity, ρ is the density, and u is the temperature. I also know *Fourier's law*

$$\boldsymbol{q}(t, \boldsymbol{x}) = -k(\boldsymbol{x})\nabla u(t, \boldsymbol{x}),$$

where k is the thermal conductivity. Putting everything together I obtain

$$c(\boldsymbol{x})\rho(\boldsymbol{x})u_t = \nabla \cdot k(\boldsymbol{x})\nabla u + f^*(t, \boldsymbol{x}), \quad t > 0, \quad x \in \Omega,$$

which is the general form of *heat* equation. Simplifying further by assuming that c, ρ, k are constants (in general the situation is opposite, in reality, c and k can depend on the temperature u itself!) I end up with

$$u_t = \alpha^2 \Delta u + f, \quad t > 0, \quad \boldsymbol{x} \in \Omega,$$

where $\alpha^2 = k/(c\rho), f = f^*/(c\rho)$ and Laplace operator $\Delta = \nabla \cdot \nabla$ (in some physics textbooks for this reason Laplace operator is often denoted as ∇^2). Clearly, even just from physical considerations, I must also have the initial and boundary conditions. For the initial conditions I have

$$u = g, \quad \text{on } \{t = 0\} \times \Omega,$$

which simply means that I know the temperature at every point of Ω at the initial time moment $t = 0$.

Boundary conditions can be in different form. First, if I assume that the temperature is fixed at the boundary then I have so-called *Dirichlet's* boundary conditions

$$u = h, \quad \text{on } \{t > 0\} \times \partial\Omega,$$

if the flux is prescribed then I have *Neumann's* boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{n}} = h, \quad \text{on } \{t > 0\} \times \partial\Omega,$$

or if the temperature of the surrounding medium is given then I have *Robin's* boundary conditions (a and b are given constants here)

$$au + b\frac{\partial u}{\partial \boldsymbol{n}} = h, \quad \text{on } \{t > 0\} \times \partial\Omega.$$

If the above classification seems to be too complicated, consider the simple case of one-dimensional heat equation

$$u_t = \alpha^2 u_{xx} + f, \quad t > 0, \quad x \in (0,1),$$

with the initial condition

$$u(0,x) = g(x), \quad x \in [0,1].$$

The Dirichlet's boundary condition becomes

$$u(t,0) = h_1(t), \quad u(t,1) = h_2(t),$$

Neumann's boundary condition is stated as

$$u_x(t,0) = h_1(t), \quad u_x(t,1) = h_2(t),$$

Robin's boundary condition is stated as

$$a_1 u(t,0) + b_1 u_x'(t,0) = h_1(t), \quad a_2 u(t,1) + b_2 u_x'(t,1) = h_2(t).$$

If all h's are zero the boundary conditions are naturally called *homogeneous*.

Exactly as in the case of one-dimensional wave equation I can use the separation of variables to solve the heat equation.

Exercise 5.7. Consider the problem

$$\begin{aligned} u_t &= u_{xx}, \quad t > 0, \quad x \in (0,1), \\ u(t,0) &= u(t,1) = 0, \quad t > 0, \\ u(0,x) &= g(x), \quad x \in [0,1]. \end{aligned}$$

Solve this problem with

1. $g(x) = 1$,
2. $g(x) = \sin 2\pi x + \frac{1}{3}\sin 4\pi x + \frac{1}{5}\sin 6\pi x$,
3. $g(x) = x - x^2$.

Sketch your solutions at different time moments.

Exercise 5.8. Solve

$$
\begin{aligned}
u_t &= u_{xx}, \quad t > 0, \quad x \in (0,1),\\
u(t,0) &= u_x(t,1) = 0, \quad t > 0,\\
u(0,x) &= x, \quad x \in [0,1].
\end{aligned}
$$

Exercise 5.9. Solve

$$
\begin{aligned}
u_t &= u_{xx}, \quad t > 0, \quad x \in (0,1),\\
u_x(t,0) &= u_x(t,1) = 0, \quad t > 0,\\
u(0,x) &= x, \quad x \in [0,1].
\end{aligned}
$$

Finally I note that assuming that f^* as well as the boundary conditions in the full heat equation do not depend on time t, it is reasonable to expect that at least in some cases the temperature will stabilize, it is said it achieves the *stationary* state. At this state $u_t = 0$ and hence u satisfies

$$\nabla \cdot k\nabla u + f^* = 0$$

with the corresponding boundary conditions. In particular, if k is constant and $f^* = 0$ I have

$$\Delta u = 0,$$

which is called the *Laplace equation.* (More general $-\Delta u = f$ is called the *Poisson* equation).

Remark 5.2. Note that one dimension Poisson equation is an ordinary differential equation

$$-\big(k(x)u'(x)\big)' = f(x),$$

with given boundary conditions, and therefore the stationary one-dimensional problems for the heat equation naturally lead to BVP for ODE.

Not surprisingly, Laplace's equation can be also solved by using separation of variables technique.

Exercise 5.10. Consider the Laplace equation on the circular disk

$$\Delta u = 0, \quad x^2 + y^2 < 1, \quad u(x,y)|_{x^2+y^2=1} = h(x,y),$$

Introducing the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ it can be proved that the Laplace operator in the polar coordinates takes the form (I do not require a proof, but this is a good exercise in the change of variables)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\,,$$

where $v(r,\theta) = u(r\cos\theta, r\sin\theta)$.

Write the boundary conditions for the equation in polar coordinates:

$$v(1,\theta) = H(\theta), \quad H(\theta) = h(x,y)|_{x^2+y^2=1}.$$

Note also that one should have $v(r,\theta) = v(r,\theta+2\pi)$, $v'(r,\theta) = v'(r,\theta+2\pi)$ (periodicity condition).

1. Separate the variables in polar coordinates and solve two ordinary differential equations, taking into account the boundary conditions. Note that additionally you should request that the solution must be bounded and smooth to satisfy the original equation. (Problem 5.5 part 4 may be of some help.)

2. Write down a formal solution to the original problem.

5.3 Motivation III: Green's functions and delta-function

In this section I consider the problem of the stationary heat distribution in a one-dimensional rod with fixed temperatures at the ends. According to the previous section my problem has the following mathematics statement:

$$-\big(k(x)u'(x)\big)' = f(x), \quad x \in (0,1),$$

with the boundary conditions $u(0) = \alpha, u(1) = \beta$. Here I assume that the conductivity of my material can be non-constant in general, and that $\alpha \neq 0, \beta \neq 0$. Function f is assumed to be continuous.

First I note that because of the linearity of my problem I can consider separately two problems

$$-\big(k(x)w'(x)\big)' = f(x), \quad x \in (0,1), \quad w(0) = w(1) = 0,$$

and

$$-\big(k(x)v'(x)\big)' = 0, \quad x \in (0,1), \quad v(0) = \alpha,\, v(1) = \beta,$$

and the solution to the original problem will be the sum $u(x) = w(x) + v(x)$, hence without loss of generality I will assume in the following that $\alpha = \beta = 0$.

I can, of course, simply integrate my equation twice and use the boundary conditions. Instead, I will use a more conceptual approach. In particular, I would like to represent my heat sources inside the rod (which are given by f) as a *linear* combination of *concentrated* sources of *unit* strength. Heuristically, if I am capable to solve my problem for each concentrated source separately, the full problem solution can be found by invoking some form of *superposition* principle. Here is my first example.

Example 5.3. Assume that $k(x) = 1$ and at the point $\xi \in (0,1)$ a concentrated source of unit strength is placed. I denote the solution of my problem as $(x,\xi) \mapsto g(x,\xi)$ emphasizing the dependence on the source position ξ. Since the source is concentrated at just one point, g must solve $-g'' = 0$ for $x \in [0,\xi)$ and $-g'' = 0$ for $x \in (\xi,1]$, with the boundary conditions $g(0,\xi) = g(1,\xi) = 0$. Hence I must have

$$\begin{aligned} g(x,\xi) &= Ax, \quad 0 \leq x < \xi, \\ g(x,\xi) &= B(1-x), \quad \xi < x \leq 1. \end{aligned}$$

Clearly I want g to be continuous at $x = \xi$, but I cannot expect it to be continuously differentiable (check that such an assumption would lead to physically meaningless $g(x,\xi) = 0$ for all x). Since my heat source has unit strength, I expect that the sum of fluxes of the heat at two points $\xi - \epsilon, \xi + \epsilon$ must be equal to 1, that is,

$$-g'|_{x=\xi+\epsilon} + g'|_{x=\xi-\epsilon} = 1,$$

where the signs reflect the simple observation that the heat flows from hotter to colder points. Taking formal limit $\epsilon \to 0$, I determine the jump of the derivative of g as

$$g'|_{x=\xi+0} - g'|_{x=\xi-0} = -1.$$

So, continuity and jump conditions together imply that

$$A\xi = B(1-\xi), \quad A + B = 1,$$

which yields

$$A = 1 - \xi, \quad B = \xi,$$

or finally

$$g(x,\xi) = \begin{cases} (1-\xi)x, & 0 \le x \le \xi, \\ (1-x)\xi, & \xi \le x \le 1. \end{cases}$$

The found function g is called *Green's* function of the differential operator $-\frac{\mathrm{d}}{\mathrm{d}x^2}$ with homogeneous Dirichlet's boundary conditions. I claim that my Green's functions allows me to solve my problem with arbitrary f.

Exercise 5.11. Consider the problem $-u'' = f$, $x \in (0,1)$ with $u(0) = u(1) = 1$. Prove that the unique solution to this problem is given by

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\, \mathrm{d}\xi,$$

where g is found above Green's function.

It is not difficult to justify the formula above when you know it. My goal here is to motivate the appearance of this formula.

Let me look back at my main example here and let me denote my concentrated source of unit strength as $\delta_\xi(x)$. By the physical reasonings δ_ξ must be zero everywhere in $x \in (0,1)$ except the point $x = \xi$ and its integral that includes the point $x = \xi$ must be equal to 1. Alas, no such function exists! So, let me motivate the nature of δ_ξ by considering a sequence of legitimate functions

$$\psi_n(x) = \begin{cases} n, & |x| < \frac{1}{2n}, \\ 0, & |x| > \frac{1}{2n}. \end{cases}$$

I would like to state something like

$$\delta_0(x) = \lim_{n \to \infty} \psi_n(x)$$

and $\delta_\xi(x) = \delta_0(x - \xi)$, but this limit (for the same reason as above) does not make much sense. The key fact is that (exercise!), if $\xi \in (a,b)$, then

$$\int_a^b \delta_\xi(x) \phi(x)\, \mathrm{d}x = \phi(\xi)$$

for any continuous at $x = \xi$ function ϕ. The introduced in this way object δ_ξ is called *delta*-function (which is not a function!) or *Dirac delta*-function, and hence I actually solved the mathematical problem

$$-g''(x,\xi) = \delta_\xi(x), \quad g(0,\xi) = g(1,\xi) = 1, \quad \xi \in (0,1),$$

where the equality must be understood in the sense of the characteristic property: If I multiply the left-hand side by any continuous ϕ and integrate, the result will be exactly $\phi(\xi)$. Now I can provide a heuristic motivation for my general solution. Indeed, from the same characteristic property I have that

$$f(x) = \int_0^1 f(\xi) \delta_\xi(x)\, \mathrm{d}\xi,$$

taking as granted "intuitively obvious" $\delta_\xi(x) = \delta_0(x-\xi) = \delta_0(\xi - x) = \delta_x(\xi)$. Now I multiply the equation $Lg = \delta_\xi$ by $f(\xi)$ and integrate from 0 to 1, obtaining

$$\int_0^1 Lg(x,\xi)f(\xi)\,\mathrm{d}\xi = \int_0^1 \delta_\xi(x)f(\xi)\,\mathrm{d}\xi = f(x).$$

Assuming that the integral and L commute, I have

$$L\left(\int_0^1 g(x,\xi)f(\xi)\,\mathrm{d}\xi\right) = f(x),$$

or, in other words,

$$u(x) = \int_0^1 g(x,\xi)f(\xi)\,\mathrm{d}\xi,$$

solves $Lu = f$ with given boundary conditions.

I do not plan to discuss the rigorous theory of objects like δ_ξ (the usual title is the theory of *generalized functions* or the theory of *distributions*), but note that with some care one can manipulate δ_ξ as a usual function. Here is a simple example.

Example 5.4. Consider again

$$-g''(x,\xi) = \delta_\xi(x), \quad g(0,\xi) = g(1,\xi) = 0$$

and integrate both sides from 0 to 1:

$$-g'(x,\xi) = H_\xi(x) + A,$$

where I made a *plausible* assumption that the integral of δ_ξ is the Heaviside function

$$H_\xi(x) = \begin{cases} 1, & x > \xi, \\ 0, & x \leq \xi. \end{cases}$$

(This also *implies* that $H'_\xi(x) = \delta_\xi(x)$ whatever this means.)

Denoting $\rho_\xi(x) = \int H_\xi(x)\,\mathrm{d}x$ the antiderivative of H_ξ (sometimes ρ is called the *ramp function*), I get

$$-g(x,\xi) = \rho_\xi(x) + Ax + B,$$

and the boundary conditions give $B = 0, A = -(1-\xi)$, thus resulting in

$$g(x,\xi) = (1-\xi)x - \rho_\xi(x),$$

which is, of course, exactly the same expression as obtained before.

I note that from the analysis above it follows that the BVP

$$u'' + \lambda u = 0, \quad u(0) = u(1) = 0$$

is *equivalent* to the integral equation

$$u(x) = \lambda \int_0^1 g(x,\xi)u(\xi)\,\mathrm{d}\xi,$$

which I will need in the following.

Example 5.5. Before finishing this section, I would like to consider Green's function for an IVP. Specifically, consider the second Newton's law

$$mu'' = F(t), \quad t > 0, u(0) = \alpha, u'(0) = \beta.$$

The Green's function associated with this problem solves (formally)

$$mg''(t,\tau) = \delta_\tau(t), \quad g(0,\tau) = g'(0,\tau) = 0.$$

The physical meaning of g is the position of a particle initially at rest subject to a unit impulse at time $t = \tau$. Similarly to the reasonings above, I must have (because the impulse will cause an instantaneous change in the momentum mu'), the jump condition

$$mg'_{t=\tau+0} - mg'|_{t=\tau-0} = 1.$$

This jump condition and continuity of g yield

$$g(t,\tau) = \begin{cases} 0, & 0 \leq t < \tau, \\ \dfrac{t-\tau}{m}, & t > \tau, \end{cases}$$

which is often called *the casual* Green's function. I will leave it as an exercise to check that the original problem has the solution

$$u(t) = \int_0^\infty g(t,\tau)F(\tau)\,\mathrm{d}\tau = \int_0^t \frac{t-\tau}{m}F(\tau)\,\mathrm{d}\tau.$$

5.4 Green's functions for BVP

All the previous discussion was intentionally non-rigorous. In this section I will present the precise mathematical statements and proofs about Green's functions.

Consider the following differential equation, where $p_0(x) \neq 0$ when $x \in [a,b]$,

$$Lu = p_0(x)\frac{\mathrm{d}^n u}{\mathrm{d}x^n} + p_1(x)\frac{\mathrm{d}^{n-1} u}{\mathrm{d}x^{n-1}} + \ldots + p_n(x)u = 0, \tag{5.1}$$

with boundary conditions specified at two boundary points a and b. Specifically, I will assume that there are n boundary conditions $U_j(u) = 0$, $j = 1, \ldots, n$, where each U_j has the form

$$U_j(u) = \alpha_{j1}u(a) + \alpha_{j2}u'(a) + \ldots + \alpha_{j,n-1}u^{n-1}(a) + \beta_{j1}u(b) + \beta_{j2}u'(b) + \ldots + \beta_{j,n-1}u^{n-1}(b) = 0. \tag{5.2}$$

Boundary conditions plus the differential equation define the *differential operator* L, which I denote using the same letter L slightly abusing the notation (to be precise: I will use the notation $Lu = 0$ for both the differential equation (5.1) alone and for the BVP (5.1), (5.2) assuming that it is clear from the context which meaning I expect to convey). My goal is to show that in some cases my operator L has an inverse, and moreover, I will built an explicit formula for my inverse.

Now to the definitions, operator B (not necessarily linear in this case) is called *inverse* with respect to operator A, if the domain D_B of B coincides with the range R_A of A and for all $x \in D_A$

$$B(Ax) = x.$$

An inverse operator is usually denoted as A^{-1}.

Exercise 5.12. Show that if A has an inverse A^{-1} then A^{-1} has an inverse and moreover $(A^{-1})^{-1} = A$.

Exercise 5.13. Show that if A is linear and A^{-1} exists then A^{-1} is linear.

Exercise 5.14. Show that linear operator A has an inverse if and only if the equation $Ax = 0$ has the unique trivial solution $x = 0$.

Now I assume that problem (5.1), (5.2) has only trivial solution $u(x) = 0$, which implies that there exists an inverse to L operator. I will show that L^{-1} is given by an integral operator with continuous kernel. This kernel is called *Green's* function of operator L.

Definition 5.6. *Consider the differential operator L defined by* (5.1), (5.2). *Green's function* $(x, \xi) \mapsto g(x, \xi)$ *of L is the function that satisfies*

1. *g is continuous and continuous differentiable with respect to x up to order $n-2$ for all x and ξ in $[a, b]$;*
2. *For each fixed $\xi \in [a, b]$ g has continuous derivatives of order $n-1$ and n with respect to x in $[a, \xi)$ and $(\xi, b]$, moreover,*

$$\frac{\partial^{n-1} g}{\partial x^{n-1}}(\xi + 0, \xi) - \frac{\partial^{n-1} g}{\partial x^{n-1}}(\xi - 0, \xi) = \frac{1}{p_0(x)};$$

3. *In each interval $[a, \xi)$, $(\xi, b]$ g as a function of x solves $Lg = 0$ and satisfies the boundary conditions $U_j(g) = 0$.*

Theorem 5.7. *If BVP $Lu = 0$ has only the trivial solution $u = 0$ then L has one and only one Green's function g.*

The proof of this theorem is constructive. That is, I present an algorithm how actually find such g.

Proof. Let $\{u_1, \ldots, u_n\}$ be a fundamental solution set to the equation $Lu = 0$ (that is, $\{u_1, \ldots, u_n\}$ is a linearly independent system of solutions). From the general theory of Chapter 3 I know that such solution sets always exist. Therefore, I must have in the interval $[a, \xi)$ that

$$g(x, \xi) = a_1 u_1(x) + \ldots + a_n u_n(x),$$

where a_j are some functions of ξ. Similarly,

$$g(x, \xi) = b_1 u_1(x) + \ldots + b_n u_n(x), \quad \xi < x \leq b.$$

From the continuity of the first $n-2$ derivatives I have

$$\big(a_1 u_1^{(j)}(\xi) + \ldots + a_n u_n^{(j)}(\xi)\big) - \big(b_1 u_1^{(j)}(\xi) + \ldots + b_n u_n^{(j)}(\xi)\big) = 0, \quad j = 0, \ldots, n-2,$$

and the jump condition takes the form

$$\big(a_1 u_1^{(n-1)}(\xi) + \ldots + a_n u_n^{(n-1)}(\xi)\big) - \big(b_1 u_1^{(n-1)}(\xi) + \ldots + b_n u_n^{(n-1)}(\xi)\big) = -\frac{1}{p_0(\xi)}\,.$$

Introducing $\boldsymbol{c}=(c_1,\ldots,c_n)^\top$, $c_j=b_j-a_j$,

$$\boldsymbol{U}=\begin{bmatrix} u_1(\xi) & \ldots & u_n(\xi)\\ \vdots & & \\ u_1^{(n-1)}(\xi) & \ldots & u_n^{(n-1)}(\xi)\end{bmatrix},$$

and $\boldsymbol{b}=(0,\ldots,0,\frac{1}{p_0(\xi)})^\top$, I can write my system succinctly as

$$\boldsymbol{U}\boldsymbol{c}=\boldsymbol{b},$$

moreover, $\det \boldsymbol{U}\neq 0$ as the Wronskian of the system evaluated at the point $x=\xi$. Hence this system has the unique solution $\boldsymbol{c}$. To determine a_j and b_j I consider the boundary conditions, which I write in the general form

$$U_j(u)=U_{ja}(u)+U_{jb}(u).$$

Then

$$U_k(g)=\sum_j a_jU_{ka}(u_j)+\sum_j b_jU_{kb}(u_j)=0.$$

Plugging $a_j=b_j-c_j$ yields

$$\sum_j b_jU_k(u_j)=\sum_j c_jU_{ka}(u_j),\quad k=1,\ldots,n,$$

which is a system of linear equations with respect to b_j whose determinant is non-zero (because I assumed that the original problem has only the trivial solution, think this out). Hence the constants a_j,b_j are determined uniquely, which proves the existence and uniqueness of Green's function. ■

Theorem 5.8. *Together with the problem $Lu=0$ consider the problem $Lu=f$ for continuous f and the same boundary conditions. If the problem $Lu=0$ has only the trivial solution then for any continuous on $[a,b]$ f problem $Lu=f$ has the unique solution given explicitly by*

$$u(x)=\int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi,$$

where g is Green's function of operator L.

Proof. To prove this theorem all I need is to check that my solution satisfies the differential equation $Lu=f$ and boundary conditions $U_j(u)=0$. Since g is $\mathcal{C}^{(n-2)}$ for all $x\in[a,b]$ I can differentiate up to the order $n-2$ under the integral sign. To evaluate $n-1$-th and n-th derivatives I must be more careful. To wit, I have

$$u^{(n-2)}(x)=\int_a^x \frac{\partial^{n-2}g}{\partial x^{n-2}}(x,\xi)f(\xi)\,\mathrm{d}\xi+\int_x^b \frac{\partial^{n-2}g}{\partial x^{n-2}}(x,\xi)f(\xi)\,\mathrm{d}\xi,$$

which I can differentiate with respect to x obtaining

$$\begin{aligned}u^{(n-1)}(x)=\int_a^x \frac{\partial^{n-1}g}{\partial x^{n-1}}(x,\xi)f(\xi)\,\mathrm{d}\xi+\frac{\partial^{n-2}g}{\partial x^{n-2}}(x,\xi)\bigg|_{\xi=x-0}f(x)+\\ \int_x^b \frac{\partial^{n-1}g}{\partial x^{n-1}}(x,\xi)f(\xi)\,\mathrm{d}\xi-\frac{\partial^{n-2}g}{\partial x^{n-2}}(x,\xi)\bigg|_{\xi=x+0}f(x)=\\ \int_a^x \frac{\partial^{n-1}g}{\partial x^{n-1}}(x,\xi)f(\xi)\,\mathrm{d}\xi+\int_x^b \frac{\partial^{n-1}g}{\partial x^{n-1}}(x,\xi)f(\xi)\,\mathrm{d}\xi.\end{aligned}$$

Similarly to the above I will find, using the jump condition for Green's function,

$$u^{(n)}(x)=\int_a^b \frac{\partial^n g}{\partial x^n}(x,\xi)f(\xi)\,\mathrm{d}\xi+\frac{1}{p_0(x)}f(x).$$

Now I plug everything into the equation $Lu=f$ and boundary conditions (5.2). For the boundary conditions I have

$$U_j(u)=\int_a^b U_j(g)f(\xi)\,\mathrm{d}\xi=0$$

since $U_j(g)=0$, so the boundary conditions hold.

For the equation I have

$$Lu=\int_a^b Lgf(\xi)\,\mathrm{d}\xi+f(x)=f(x),$$

which proves that u satisfies the equation. The uniqueness proof is left as an exercise. ■

Remark 5.9. Operator A, which is defined as

$$Af(x)=\int_a^b K(x,\xi)f(\xi)\,\mathrm{d}\xi,$$

is called an integral operator with kernel K. The proven theorem thus means that for the differential operator L the inverse L^{-1} is an integral operator with kernel g.

Remark 5.10. Quite often (see the motivational sections, for instance) one has to consider BVP of the form

$$Lu=\lambda u+f,\quad U_j(u)=0,\quad j=1,\ldots,n. \tag{5.3}$$

Assume that differential operator L has an inverse L^{-1} with the corresponding Green's function g. Then I proved that problem (5.3) is equivalent to the integral equation

$$u(x)=\lambda\int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi+h(x),\quad h(x)=\int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi.$$

Example 5.11. Here is one explicit example of a computation of Green's function for BVP. Let

$$L=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\omega^2,\quad \omega>0,\quad u(0)=u(1)=0.$$

It is convenient to express the general solution to the homogeneous equation as

$$A\sinh\omega x+B\cosh\omega x.$$

(Here the idea is to use such basis functions that satisfy the initial conditions $u(0)=1,u'(0)=0$ and $u(0)=0$, $u'(0)=1$, recall the notion of the principal matrix solution). Thus I have

$$g(x,\xi)=\begin{cases}C_1\sinh\omega x+C_2\cosh\omega x, & x\leq\xi,\\ C_3\sinh\omega x+C_4\cosh\omega x, & x\geq\xi.\end{cases}$$

From the boundary conditions I find

$$g(x,\xi)=\begin{cases}C_1\sinh\omega x, & x\leq\xi\\ C_3\sinh\omega(1-x), & x\geq\xi.\end{cases}$$

The second line can be inferred directly, but also can be argued as follows. From the second boundary condition I have that

$$C_3=-\frac{C_4\cosh\omega}{\sinh\omega}.$$

Therefore,

$$\frac{C_4}{\sinh\omega}(\sinh\omega\cosh\omega x-\cosh\omega\sinh\omega x)=A\sinh\omega(1-x),$$

where I used

$$\sinh(\alpha\pm\beta)=\sinh\alpha\cosh\beta\pm\sinh\beta\cosh\alpha.$$

Now the continuity and jump conditions becomes

$$C_1\sinh\omega\xi=C_3\sinh\omega(1-\xi),$$
$$-\omega C_3\cosh\omega(1-\xi)-C_1\omega\cosh\omega\xi=-1.$$

Multiplying the first equation by $\omega\cosh\omega(1-\xi)$ and the second by $\sinh\omega(1-\xi)$ and adding, I find, after again using the formula for $\sinh(\alpha+\beta)$, that

$$C_1=\frac{\sinh\omega(1-\xi)}{\omega\sinh\omega},\quad C_3=\frac{\sinh\omega\xi}{\omega\sinh\omega}.$$

Finally,

$$g(x,\xi)=\begin{cases}\dfrac{\sinh\omega x\sinh\omega(1-\xi)}{\omega\sinh\omega}, & x\leq\xi,\\[2mm] \dfrac{\sinh\omega(1-x)\sinh\omega\xi}{\omega\sinh\omega}, & x\geq\xi.\end{cases}$$

Exercise 5.15. Find Green's function for

$$u''+u=f(x),\quad u(0)=u(\pi),\quad u'(0)=u'(\pi).$$

Exercise 5.16. Find Green's function for

$$x^3u''+3x^2u'+xu=f(x),\quad u(1)=0,u(2)+2u'(2)=0.$$

Exercise 5.17. Find Green's function for

$$u''\cos^2x-u'\sin 2x=f(x),\quad u(0)=u'(0),\quad u(\pi/4)+u'(\pi/4)=0.$$

Exercise 5.18. For which a there exist Green's function for BVP

$$u''+au=f(x),\quad u(0)=u(1)=0?$$

Exercise 5.19. Find Green's function for the stationary problem with variable conductivity

$$-\big(k(x)u'(x)\big)' = f(x), \quad u(0) = 0,\, u'(1) = 0,$$

where the physical meaning for the boundary conditions that the left-hand side of the rod has temperature zero and the right-hand side is insolated.

Exercise 5.20. The small transverse deflection u of a homogeneous beam of unit length subject to a distributed transverse loading f satisfies

$$u^{(4)} = f(x), \quad 0 < x < 1,$$

with boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

Find Green's function for this differential operator and solve this problem for continuous f.

5.5 Inner products and Hilbert spaces

I already discussed a number of properties of various BVP, which may seem somewhat unrelated. The actual unifying view is obtained through the use of the general abstract language of *Hilbert spaces*, which I carefully define and discuss in this and the next sections.

Definition 5.12. *Let X be a complex vector space. An inner product on X is a function*

$$\langle\cdot\,,\cdot\rangle : X \times X \longrightarrow \mathbf{C},$$

for which the following properties hold for any $x, y, z \in X$ and $\alpha \in \mathbf{C}$:

$$\begin{aligned}
&(1) \quad \langle x + y\,, z\rangle = \langle x\,, z\rangle + \langle y\,, z\rangle \\
&(2) \quad \langle \alpha x\,, y\rangle = \alpha\,\langle x\,, y\rangle \\
&(3) \quad \langle x\,, y\rangle = \overline{\langle y\,, x\rangle} \\
&(4) \quad \langle x\,, x\rangle > 0 \quad \textit{if } x \neq 0.
\end{aligned}$$

The immediate consequences are

$$\begin{aligned}
&(5) \quad \langle x\,, y + z\rangle = \langle x\,, y\rangle + \langle x\,, z\rangle \\
&(6) \quad \langle x\,, \alpha y\rangle = \overline{\alpha}\,\langle x\,, y\rangle \\
&(7) \quad \langle 0\,, x\rangle = \langle x\,, 0\rangle = \langle 0\,, 0\rangle = 0.
\end{aligned}$$

The properties (3) and (6) look somewhat easier if X is a real vector space. In the following, if not stated otherwise, I will assume that X is a complex vector space. Some proofs may be different for real and complex cases. The student should remember that some properties are *different* for real and complex vector spaces.

Exercise 5.21. Let A be a linear operator on X with inner product $\langle\cdot\,,\cdot\rangle$. Show that if X is complex and $\langle Ax\,, x\rangle = 0$ for all $x \in X$ then $A = 0$ whereas this is not true for real vector space X.

The properties of the inner product are motivated by the real dot product

$$\boldsymbol{x}\cdot\boldsymbol{y}=\langle\boldsymbol{x}\,,\boldsymbol{y}\rangle=\sum_{j=1}^{3}x_jy_j,\quad \boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^3,$$

which is actually equivalent to the dot product

$$\boldsymbol{x}\cdot\boldsymbol{y}=|\boldsymbol{x}||\boldsymbol{y}|\cos\theta,$$

where θ is an angle between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and hence allows to measure angles. The complex conjugation is necessary to guarantee that if $\boldsymbol{x}\in\mathbf{C}^3$ then

$$\boldsymbol{x}\cdot\boldsymbol{x}\in\mathbf{R}.$$

The definition I chose means that for any $\boldsymbol{x}$, $\boldsymbol{y}\in\mathbf{C}^3$

$$\boldsymbol{x}\cdot\boldsymbol{y}=\sum_{j=1}^{3}x_j\overline{y}_j$$

is taken as the standard inner product. Sometimes (in particular, often in Physics) an expression is used when the conjugation is applied to the first factors.

Definition 5.13. *Pair $(X,\langle\cdot\,,\cdot\rangle)$ is an inner product space.*

Each inner product induces a norm by the formula

$$\|x\|:=\langle x\,,x\rangle^{1/2}.$$

Lemma 5.14 (Cauchy–Bunyakovskii–Schwarz inequality)**.**

$$|\langle x\,,y\rangle|\leq\|x\|\|y\|$$

Proof. I will assume that X is a real vector space. The case of the complex vector space is left as an exercise. For any real constant α

$$0\leq\langle x+\alpha y\,,x+\alpha y\rangle=\|x\|^2+2\alpha\langle x\,,y\rangle+\alpha^2\|y\|^2,$$

which is a quadratic polynomial with respect to the variable α, and hence should have the discriminant non-positive:

$$\langle x\,,y\rangle^2-\|x\|^2\|y\|^2\leq 0,$$

which concludes the proof. ∎

The lemma also shows that the equality is possible if and only if x and y are linearly dependent (prove it), which in its turn implies that

$$\|x\|=\sup_{y\neq 0}\frac{|\langle x\,,y\rangle|}{\|y\|},$$

since
$$\|x\| = \frac{\langle x\,, x\rangle}{\|x\|} \le \sup_{y\neq 0} \frac{|\,\langle x\,, y\rangle\,|}{\|y\|} \le \|x\|.$$
Moreover, now one can easily show that $\|\cdot\|$ is indeed a norm. Consider the expression
$$\|x+y\|^2 = \langle x+y\,, x+y\rangle \le \|x\|^2 + 2|\,\langle x\,, y\rangle\,| + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$
which proves the triangle inequality, the other properties of the norm are straightforward.

Having a norm on a vector space X means that we have a metric and a topology induces by this metric, and therefore we can talk about convergence.

Definition 5.15. *An inner product space is called a Hilbert space if it is complete (with respect to the norm induced by the inner product).*

The natural question whether any norm can be generated by some inner product. The answer is negative, and the simplest way to check whether $\|\cdot\|$ can be generated by an inner product is to check the parallelogram law:

Lemma 5.16 (Parallelogram law)**.** *If X is an inner product space then*
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$
for any $x, y \in X$.

The proof is left as an (easy) exercise.

Exercise 5.22. Assume that X is a normed vector space whose norm satisfies the parallelogram law. Show that one can consider X to be an inner product space with a suitably defined inner product.

Example 5.17. The real vector space $\mathbf{R}^k$ becomes a (finite dimensional) Hilbert space if the inner product is defined as
$$\langle \boldsymbol{x}\,, \boldsymbol{y}\rangle_{\boldsymbol{A}} = \sum_{i,j=1}^{k} a_{ij} x_i y_j$$
for any positive definite symmetric real matrix $\boldsymbol{A} = (a_{ij})_{k\times k}$. The complex vector space $\mathbf{C}^k$ becomes a (finite dimensional) Hilbert space if the inner product is defined as
$$\langle \boldsymbol{x}\,, \boldsymbol{y}\rangle_{\boldsymbol{A}} = \sum_{i,j=1}^{k} a_{ij} x_i \overline{y}_j$$
for any positive definite Hermitian matrix $\boldsymbol{A}$.

Example 5.18. The complex space l^2 consists of all infinite sequences $x = (x_i)_{i=1}^{\infty}$ that satisfy $\sum_{i=1}^{\infty} |x_i|^2 < \infty$ (square summable sequences) with the inner product
$$\langle x\,, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y}_i.$$

The inner product is well defined due to the Cauchy–Bunyakovskii–Schwarz inequality.

The first thing to note about l^2 is that it is not finite dimensional, opposite to $\mathbf{R}^k$ or $\mathbf{C}^k$ and therefore is infinite dimensional. It is a Hilbert space, because every Cauchy sequence has a limit. Moreover, there exists a countable dense subset of l^2, which means that any element of l^2 is the limit of elements from this subset (can you construct this subset explicitly?) Such metric spaces, containing finite or countable dense subset are called *separable*. In what follows I will consider only *separable* Hilbert spaces, which makes the discussion much easier (and, moreover, all the spaces I will need to deal with are separable).

Example 5.19. Consider the space of continuous real function $\mathcal{C}([a, b]; \mathbf{R})$ with the inner product

$$\langle x\,, y\rangle = \int_a^b x(t)y(t)\,\mathrm{d}t.$$

You should check that it is indeed an inner product. This inner product space is not Hilbert because it is quite easy to give examples of sequences of continuous functions which would lead to still integrable but discontinuous functions. Fortunately, it is always possible to consider a *completion* of an inner product space, and the completion of the space of continuous functions, defined on the set Ω is denoted $\mathcal{L}^2(\Omega)$ (I use this particular font to distinguish the space from differential operator L, in many book the notation $L^2(\Omega)$ is used).

More precisely, $\mathcal{L}^2(\Omega)$ is a Hilbert space of all (equivalence classes of) measurable functions x that are square summable, i.e.,

$$\int_\Omega |x|^2 < \infty,$$

where the integral is Lebesgue's integral. The inner product is given by (in real case)

$$\langle x\,, y\rangle = \int_\Omega xy,$$

and

$$\langle x\,, y\rangle = \int_\Omega x\bar{y},$$

in the complex case. The corresponding norm

$$\|x\|_{\mathcal{L}^2(\Omega)} = \left(\int_\Omega |x|^2\right)^{1/2}.$$

The Hilbert space $\mathcal{L}^2(\Omega)$ is separable (as well as all other $\mathcal{L}^p$ spaces for $1 \leq p < \infty$; the space $\mathcal{L}^\infty$ is not separable).

5.6 Orthogonality. Basis. Fourier series

In inner product spaces we can measure angles between vectors. In particular, we can specify which two vectors are orthogonal.

Definition 5.20. *Let $(X, \langle\cdot\,,\cdot\rangle)$ be an inner product space. Vectors $x, y \in X$ are orthogonal if*

$$\langle x\,, y\rangle = 0.$$

An immediate consequence of this definition is the famous Pythagoras theorem:

Lemma 5.21 (Pythagoras theorem). *If x and y are orthogonal then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Given any family of vectors $(e_i)_{i\in I}$, $\operatorname{span}(e_i)_{i\in I}$ denotes all the subspace of X formed by all finite linear combinations in this family. A family $(e_i)_{i\in I}$ of elements of an inner product space $(X, \langle\cdot\,,\cdot\rangle)$ is called *orthonormal family* if

$$\langle e_i\,, e_j\rangle = \delta_{ij}, \quad \text{for all } i, j \in I.$$

Here δ_{ij} is Kronecker's symbol, which is equal to 1 is $i = j$ and 0 otherwise. Note that any orthonormal family is linearly independent (prove it). Gram–Schmidt orthonormalization procedure allows to construct an orthonormal family out of linearly independent (finite or countably infinite) family of vectors. Details are left to the textbooks on functional analysis.

An orthonormal family $(e_i)_{i\in I}$ in an inner product space $(X, \langle\cdot\,,\cdot\rangle)$ is called *maximal* if the only vector $x \in X$ that satisfies $\langle x\,, e_i\rangle = 0$ for all $i \in I$ is $x = 0$. A convenient sufficient condition for an orthonormal family to be maximal is as follows: $(e_i)_{i=I}$ is maximal if

$$\overline{\operatorname{span}(e_i)_{i\in I}} = X,$$

where the bar denotes the closure.

Any inner product space has a maximal orthonormal family, but in general the proof will require the axiom of choice. However, in the case of a separable inner product space, more can be said without any heavy machinery. In particular,

Theorem 5.22. *Let $(X, \langle\cdot\,,\cdot\rangle)$ be a separable infinite dimensional inner product space. Then*

(a) There exists a countable infinite orthonormal family $(e_i)_{i=1}^{\infty}$ such that $x \in X$ and $\langle x\,, e_i\rangle = 0$ for all i imply that $x = 0$.

(b) Any orthonormal family is finite or countably infinite.

Exercise 5.23. I will leave a proof of this theorem as an exercise.

Finally, I will need definition of a basis for an infinite dimensional space.

Definition 5.23. *Let X be a Banach space. Then a countably infinite family $(e_i)_{i=1}^{\infty}$ is a (Schauder) basis for X if for any $x \in X$ there exist unique scalars $(\alpha_i)_{i=1}^{\infty}$ such that*

$$x = \sum_{i=1}^{\infty} \alpha_i e_i.$$

If the discussion above was clear, it should not come as a surprise that in a separable Hilbert space a maximal orthonormal family is exactly a (Schauder) basis. To practice working with elements of Hilbert space, I will prove

Theorem 5.24. *Let $(X, \langle\cdot\,,\cdot\rangle)$ be an infinite dimensional separable Hilbert space and let $(e_i)_{i=1}^{\infty}$ be a maximal orthonormal family. Then*

(a) *Any element $x \in X$ can be expanded as a convergent series*

$$x = \sum_{i=1}^{\infty} \langle x\,, e_i\rangle\, e_i,$$

which is called the Fourier series of x.

(b) *The scalars $\langle x\,, e_i\rangle \in \mathbf{C}$, $i \geq 1$, which are called the Fourier coefficients of x relative to the family $(e_i)_{i=1}^{\infty}$, satisfy Parseval's formula*

$$\|x\|^2 = \langle x\,, x\rangle = \sum_{i=1}^{\infty} |\,\langle x\,, e_i\rangle\,|^2.$$

Proof. First, let me show that $\sum_{i=1}^{\infty} |\,\langle x\,, e_i\rangle\,| < \infty$. We have

$$\begin{aligned}
0 \leq \|x - \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i\|^2 &= \left\langle x - \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i\,, x - \sum_{j=1}^{n} \langle x\,, e_j\rangle\, e_j \right\rangle = \\
&= \|x\|^2 - \sum_{i=1}^{n} |\,\langle x\,, e_i\rangle\,|^2 - \sum_{j=1}^{n} |\,\langle x\,, e_i\rangle\,|^2 + \sum_{i,j=1}^{n} \langle x\,, e_i\rangle\, \overline{\langle x\,, e_j\rangle}\, \langle e_i\,, e_j\rangle = \\
&= \|x\|^2 - \sum_{i=1}^{n} |\,\langle x\,, e_i\rangle\,|^2.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} |\,\langle x\,, e_i\rangle\,|^2 \leq \|x\|^2, \quad \text{for any } n,$$

and hence

$$\sum_{i=1}^{\infty} |\,\langle x\,, e_i\rangle\,|^2 \leq \|x\|^2.$$

The last inequality is called *Bessel's inequality.* Note that I did not use the fact that my space is Hilbert. This inequality is true in any inner product space.

Next, I will show $\sum_{i=1}^{\infty} \langle x\,, e_i\rangle\, e_i$ converges. Consider the sequence of partial sums

$$s_n = \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i,$$

and

$$\|s_n - s_l\|^2 = \left\langle \sum_{i=l+1}^{n} \langle x\,, e_i\rangle\, e_i\,, \sum_{j=l+1}^{n} \langle x\,, e_j\rangle\, e_j \right\rangle = \sum_{i=l+1}^{n} |\,\langle x\,, e_i\rangle\,|^2,$$

which proves that (s_n) is a Cauchy sequence. Since the space is complete (Hilbert), then there is the limit $y := \sum_{i=1}^{\infty} \langle x\,, e_i\rangle\, e_i$.

Finally, I need to show that $x = y$. It is equivalent to consider the condition $\langle x - y\,, e_i\rangle$ for all $i \geq 1$ since $(e_i)_{i=1}^{\infty}$ is maximal by the assumptions. Using the continuity of the inner product, I have

$$\left\langle x - \sum_{i=1}^{\infty} \langle x\,, e_i\rangle\, e_i\,, e_j \right\rangle = \lim_{n\to\infty} \left\langle x - \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i\,, e_j \right\rangle.$$

But

$$\left\langle x - \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i\,, e_j \right\rangle = 0, \quad \text{for all } n \geq j.$$

Therefore, $x = y$.

Finally, from the previous I established that

$$0 = \lim_{n\to\infty} \|x - \sum_{i=1}^{n} \langle x\,, e_i\rangle\, e_i\|^2 = \lim_{n\to\infty} \left(\|x\|^2 - \sum_{i=1}^{n} |\,\langle x\,, e_i\rangle\,|^2 \right),$$

which proves *Parseval's formula*. ■

Example 5.25. Consider the space $\mathcal{L}^2([0, 2\pi]; \mathbf{C})$ and the family, defined by

$$e_n(\theta) := \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}n\theta}, \quad n \in \mathbf{Z}, \quad 0 \leq \theta \leq 2\pi.$$

Note that the index n takes both positive and negative values. This is an orthonormal family in the Hilbert space with the inner product

$$\langle x\,, y\rangle = \int_0^{2\pi} x(\theta)\overline{y(\theta)}\,\mathrm{d}\theta,$$

which can be checked directly.

To prove that it is a maximal family requires several additional facts, in particular it can be done by using a complex version of the Weierstrass trigonometric polynomial approximation theorem. Let me state this fact for now without a proof. The proved theorem asserts that the family $(e_n)_{n\in\mathbf{Z}}$ is a basis for our Hilbert space, and any element of this space can be uniquely represented by its Fourier coefficients.

The last, and probably most important question to ask is what is an ultimate source of maximal orthonormal families (note that the maximality is usually the property, which is most difficult to prove).

5.7 Self-adjoint operators

Recall that an operator L from a normed vector space X to a normed vector space Y is linear if

$$L(\alpha x + y) = \alpha Lx + Ly, \quad \text{for any } x, y \in X.$$

A linear operator is bounded if there is a constant $C \geq 0$ such that $\|Lx\| \leq C\|x\|$ for any $x \in X$. A basic fact that a linear operator is bounded if and only if it is continuous. For continuous (and, correspondingly, bounded linear operators) it is convenient to introduce the operator (or uniform) norm, which is given by the infimum among all possible constants C. Recall that

$$\|L\| = \sup_{x\neq 0} \frac{\|Lx\|}{\|x\|} = \sup_{\|x\|=1} \|Lx\| = \sup_{\|x\|\leq 1} \|Lx\|.$$

Assume now that we deal with an inner product space $(X, \langle \cdot\,, \cdot\rangle)$.

Definition 5.26. *A linear operator $L\colon X \longrightarrow X$ is called self-adjoint if*

$$\langle Lx\,, y\rangle = \langle x\,, Ly\rangle\,, \quad \text{for all } x, y \in X.$$

It can be shown that if X is a Hilbert space then any self-adjoint operator is continuous and hence bounded.

For any linear operator L it is possible to consider an eigenvalue problem: To find all scalars λ and non-zero vectors $x \in X$ that satisfy the relation $Lx = \lambda x$. The scalars are called eigenvalues and the vectors are eigenvectors. This is a direct generalization of the eigenvalue problem on finite dimensional vector spaces.

For the self-adjoint operators I have

Theorem 5.27. *Let L be a self-adjoint operator on inner product space X. Then*

(a) *$\langle Lx\,, x\rangle$ is real for any $x \in X$.*

(b) *Let λ be an eigenvalue of L. Then λ is real. Moreover, $\lambda \geq 0$ if L is positive semi-definite and $\lambda > 0$ if L is positive definite.*

(c) *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*

(d)

$$\|L\| = \sup_{x\neq 0,\, y\neq 0} \frac{|\,\langle Lx\,, y\rangle\,|}{\|x\|\|y\|} = \sup_{x\neq 0} \frac{|\,\langle Lx\,, x\rangle\,|}{\|x\|^2} = \sup_{\|x\|=1} |\,\langle Lx\,, x\rangle\,|.$$

Proof. (a) $\langle Lx\,, x\rangle = \overline{\langle x\,, Lx\rangle} = \overline{\langle Lx\,, x\rangle}$.

(b) Let $Lx = \lambda x$ and $x \neq 0$. Then $\langle Lx\,, x\rangle = \lambda \langle x\,, x\rangle$, and

$$\lambda = \frac{\langle Lx\,, x\rangle}{\langle x\,, x\rangle},$$

which, by using (a), proves (b).

(c) Let $Lx_1 = \lambda_1 x_1$ and $Lx_2 = \lambda_2 x_2$ such that $\lambda_1 \neq \lambda_2$. I have

$$\langle Lx_1\,, x_2\rangle = \lambda_1 \langle x_1\,, x_2\rangle = \langle x_1\,, Lx_2\rangle = \lambda_2 \langle x_1\,, x_2\rangle \implies (\lambda_1 - \lambda_2)\langle x_1\,, x_2\rangle = 0.$$

(d) Let me start with the first expression for $\|L\|$. Let

$$\alpha = \sup_{x\neq 0, y\neq 0} \frac{|\,\langle Lx\,, y\rangle\,|}{\|x\|\|y\|} = \sup_{\|x\|=\|y\|=1} |\,\langle Lx\,, y\rangle\,|\,.$$

To prove the formula, I will show that $\alpha \leq \|L\| \leq \alpha$. The first inequality is a consequence of the Cauchy–Bunyakovskii–Schwarz inequality, since

$$|\,\langle Lx\,, y\rangle\,| \leq \|Lx\|\|y\| \leq \|L\|\|x\|\|y\|.$$

To prove the second inequality, start with $\|x\| = 1$

$$\|Lx\|^2 = \langle Lx\,, Lx\rangle = \|Lx\|\frac{\langle Lx\,, Lx\rangle}{\|Lx\|} \leq \|Lx\| \sup_{\|x\|=1, \|y\|=1} |\,\langle Lx\,, y\rangle\,| = \|Lx\|\alpha,$$

which, using

$$\|L\| = \sup_{\|x\|=1} \|Lx\| \leq \alpha,$$

finishes the proof.

Now to the second part of (d), for which I introduce

$$\beta = \sup_{x\neq 0} \frac{|\langle Lx\,,x\rangle|}{\|x\|^2} = \sup_{\|x\|=1} |\langle Lx\,,x\rangle|.$$

The goal is to show that $\beta \leq \|L\| \leq \beta$. The first follows from the Cauchy–Bunyakovskii–Schwarz inequality. For the second inequality I first note that

$$\langle Lx\,,x\rangle \leq \beta\|x\|^2.$$

Now, direct calculation together with the parallelogram law for two arbitrary $x, y \in X$ for which $\|x\| = \|y\| = 1$, yield

$$\begin{aligned} 4\,\mathrm{Re}\,\langle Lx\,,y\rangle &= \langle L(x+y)\,,x+y\rangle - \langle L(x-y)\,,x-y\rangle \\ &\leq \beta\left(\|x+y\|^2 + \|x-y\|^2\right) = 2\beta\left(\|x\|^2+\|y\|^2\right) = 4\beta, \end{aligned}$$

or

$$\mathrm{Re}\,\langle Lx\,,y\rangle \leq \beta.$$

Now,

$$\langle Lx\,,y\rangle = |\langle Lx\,,y\rangle|e^{\mathrm{i}\theta} = \left\langle Lx\,,ye^{-\mathrm{i}\theta}\right\rangle.$$

Since $\|ye^{-\mathrm{i}\theta}\| = 1$ then

$$\mathrm{Re}\left\langle Lx\,,ye^{-\mathrm{i}\theta}\right\rangle = \mathrm{Re}\,e^{\mathrm{i}\theta}|\langle Lx\,,y\rangle| = |\langle Lx\,,y\rangle|,$$

which implies

$$\|L\| = \sup_{\|x\|=\|y\|=1} |\langle Lx\,,y\rangle| \leq \beta.$$

■

Exercise 5.24. Show that the integral operator

$$Au(x) = \int_a^b g(x,\xi)u(\xi)\,\mathrm{d}\xi$$

is self-adjoint on $\mathcal{L}^2([a,b];\mathbf{C})$ if and only if $g(x,\xi) = \overline{g(\xi,x)}$.

The big question in the last theorem is why I needed at all the last point (d), since it seems that the properties of the eigenvalues and eigenvectors of a self-adjoint operator are almost what I am looking for. The answer is that I never proved that eigenvalues of L must exist at all, let alone form a countable set. Actually, how simple examples show, it is possible to come up with self-adjoint operators that do not have eigenvalues at all, or have an uncountable number of eigenvalues. To remedy this issue I will have to choose only those self-adjoint operators, which are at the same time *compact*.

Exercise 5.25. Consider an operator M mapping $\mathcal{L}^2[0,1]$ to $\mathcal{L}^2[0,1]$ defined by

$$y(t) = tx(t)$$

(multiplication by t). Show that M is self-adjoint. Also show that M has no eigenvalues.

5.8 Compact self-adjoint operators. Spectral theorem

Definition 5.28. *A linear operator $A\colon X \longrightarrow Y$ between normed vector spaces X and Y is compact if for any bounded sequence $(x_n)_{n=1}^{\infty}$ in X from the sequence (Ax_n) in Y it is possible to choose a convergent subsequence.*

If an operator is compact than it is bounded and hence continuous.

To give an example of a compact operator I need a tool to determine when my operator is compact. In some cases it is almost immediate. For instance, convince yourself that if the range of A is finite dimensional then A is compact. Here is one more useful criterion.

Lemma 5.29. *Let $A\colon X \longrightarrow Y$ be a linear operator and let (A_n) be a sequence of compact operators. If $A_n \to A$ in the operator norm, then A is compact.*

Proof. ∎

Example 5.30. Consider the integral operator

$$Au(x) = \int_{\Omega} K(x,\xi)u(\xi)\,\mathrm{d}\xi,$$

with integrable kernel K. It is a compact operator from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)$.

Proof. ∎

It is a remarkable fact that any compact and self-adjoint operator acting in an infinite dimensional inner product space possesses similar properties. In particular, there are at most countably infinite number of nonzero real eigenvalues, each one of finite multiplicity, and the corresponding eigenvectors form a maximal orthonormal family if the operator is in addition injective.

Theorem 5.31. *Let $(X, \langle\cdot\,,\cdot\rangle)$ be an infinite dimensional inner product space and let $A\colon X \longrightarrow X$ be a compact and self-adjoint operator with an infinite dimensional range. Then*[2]

(*a*) *There exists an infinite sequence (λ_n) of eigenvalues of A and an infinite sequence (p_n) of corresponding eigenvectors that satisfy*

$$\begin{aligned}
&|\lambda_1| = \|A\|, \quad |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq \ldots, \quad \lambda_n \neq 0, \quad \lim_{n\to\infty}\lambda_n = 0,\\
&Ap_n = \lambda_n p_n, \quad \langle p_k\,, p_l\rangle = \delta_{kl}, \quad k,l \geq 1,\\
&|\lambda_1| = |\,\langle Ap_1\,, p_1\rangle\,| = \sup_{x\neq 0}\frac{|\,\langle Ax\,, x\rangle\,|}{\|x\|^2},\\
&|\lambda_n| = |\,\langle Ap_n\,, p_n\rangle\,| = \sup_{x\neq 0, \langle x\,, p_k\rangle = 0,\, 1\leq k\leq n-1}\frac{|\,\langle Ax\,, x\rangle\,|}{\|x\|^2}, \quad n \geq 2.
\end{aligned}$$

(*b*) *For any vector $x \in X$,*

$$Ax = \sum_{n=1}^{\infty}\lambda_n\,\langle x\,, p_n\rangle\, p_n.$$

[2] I am copying a very detailed proof from Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM, 2013.

(*c*) *Let* λ *be any nonzero eigenvalue of* A. *Then there exists* $n \geq 1$ *such that* $\lambda_n = \lambda$, *moreover the set of such* λ_n *is finite.*

(*d*) *If* L *is injective* (*i.e.,* $\ker L = \{0\}$) *then* (p_n) *is a maximal orthonormal family, and hence a basis if* X *is a separable Hilbert space.*

Proof. (*i*) First, I will prove that there exist λ_1 and p_1, $\|p_1\| = 1$ such that $0 < |\lambda_1| = \|A\| = |\langle Ap_1\,, p_1\rangle|$ and $Ap_1 = \lambda_1 p_1$.

Recall that $\|A\| = \sup_{\|x\|=1} |\langle Ax\,, x\rangle|$ and $\|A\| > 0$ due to the assumption that the range of A is infinite dimensional. Therefore, there exists a sequence (x_n) such that $|\langle Ax_n\,, x_n\rangle| \to \|A\|$ with $\|x_n\| = 1$ for all n. Introduce $|\lambda_1| = \lim_{n\to\infty} |\langle Ax_n\,, x_n\rangle| = \|A\| > 0$. Since A is compact that from the sequence (Ax_n) it is possible to choose a convergent subsequence, say (Ax_l), which converges in X.

Consider (recalling that $\langle Ax_l\,, x_l\rangle$ is real even in the complex case)

$$\begin{aligned}\|Ax_l - \lambda_1 x_l\|^2 = \|Ax_l\|^2 - 2\lambda_1 \langle Ax_l\,, x_l\rangle + \lambda_1^2 \leq \\ \leq \|A\|^2 - 2\lambda_1 \langle Ax_l\,, x_l\rangle + \lambda_1^2 \to 0, \quad l \to \infty.\end{aligned}$$

Therefore $Ax_l - \lambda_1 x_l \to 0$ when $l \to \infty$.

Define $p_1 := \lim_{l\to\infty} x_l$. Then from the above and rearrangements

$$p_1 = \lim_{l\to\infty} x_l = \lim_{l\to\infty} \left(-\frac{1}{\lambda_1}(Ax_l - \lambda_1 x_l) + \frac{1}{\lambda_1} Ax_l \right) = \frac{1}{\lambda_1} \lim_{l\to\infty} Ax_l,$$

and $\|p_1\| = 1$ since $\|x_l\| = 1$ for all l.

Now

$$Ap_1 = A(\lim_{l\to\infty} x_l) = \lim_{l\to\infty} Ax_l = \lambda_1 p_1,$$

which proves that $\lambda_1 = \|A\|$ or $\lambda_1 = -\|A\|$ is an eigenvalue of A with unit eigenvector p_1.

(*ii*) Now let me make a second step. Consider the subspace $X_2 := \{x \in X \colon \langle x\,, p_1\rangle = 0\}$. Then the image of $A(X_2)$ of X_2 is again in X_2, since

$$\langle Ax\,, p_1\rangle = \langle x\,, Ap_1\rangle = \lambda_1 \langle x\,, p_1\rangle = 0.$$

Therefore, I can consider the restriction A_2 of A to X_2, which is again compact and self-adjoint operator. Moreover, $A_2 \neq 0$. Indeed, assuming that $A_2 = 0$ then

$$A(x - \langle x\,, p_1\rangle p_1) = 0, \quad x \in X,$$

because $x - \langle x\,, p_1\rangle p_1 \in X_2$. On the other hand, I find

$$Ax = \lambda_1 \langle x\,, p_1\rangle p_1$$

for all $x \in X$, which contradicts the fact that the image of A is infinite dimensional. Now the arguments from (*i*) are repeated verbatim for A_2, and we find that there exists λ_2 and p_2, $\|p_2\| = 1$, $Ap_2 = \lambda_2 p_2$, $\langle p_1\,, p_2\rangle = 0$,

$$0 \leq |\lambda_2| = \|A_2\| = \sup_{x\neq 0,\, x\in X_2} \frac{|\langle A_2 x\,, x\rangle|}{\|x\|^2} = \sup_{x\neq 0,\, x\in X_2} \frac{|\langle Ax\,, x\rangle|}{\|x\|^2} \leq \sup_{x\neq 0} \frac{|\langle Ax\,, x\rangle|}{\|x\|^2} = |\lambda_1|.$$

We can continue. The only thing to check every step that $A_{n+1}\neq 0$. This can be argued similarly, by considering $x-\sum_{m=1}^{n}\langle x\,,p_m\rangle\,p_m$, which belongs to X_{n+1}, and assuming that $A_{n+1}=0$, thus getting a contradiction.

To summarize I showed that compact and self-adjoint operator A with infinite dimensional range always has infinite sequence of eigenvalues (λ_n) and eigenvectors (p_n), and moreover $|\lambda_1|=\|A\|\geq|\lambda_2|\geq\ldots$.

(iii) Now let me show that $\lim_{n\to\infty}\lambda_n=0$. Assume that it is not true, i.e., $|\lambda_n|\geq\delta>0$ for all n. Now consider a sequence $(\lambda_n^{-1}p_n)$, which is bounded and therefore has a convergent subsequence, say $(\lambda_m^{-1}p_m)$. Since A is compact, I can find yet another convergent subsequence $(A\lambda_l^{-1}p_l)=(p_l)$. But since all p_n are orthogonal then

$$\|p_k-p_l\|^2=\|p_k\|^2+\|p_l\|^2=2,$$

and hence I cannot have the limit. Contradiction. Part (a) of the theorem is done.

(iv) Consider $x_n:=x-\sum_{k=1}^{n}\langle x\,,p_k\rangle\,p_k$. By construction, and using the notations introduced above, I have that $x_n\in X_{n+1}=\{x\in X\colon\langle x\,,p_k\rangle=0,\,1\leq k\leq n\}$. Since x_n is orthogonal to $\sum_{k=1}^{n}\langle x\,,p_k\rangle\,p_k$, then, by the Pythagoras theorem, $\|x_n\|\leq\|x\|$. Moreover, $Ax_n=A_{n+1}x_n$, and $\|A_{n+1}\|=|\lambda_{n+1}|$.

Now

$$\|Ax-\sum_{k=1}^{n}\lambda_k\,\langle x\,,p_k\rangle\,p_k\|=\|Ax_n\|=\|A_{n+1}x_n\|\leq\|A_{n+1}\|\|x_n\|\leq|\lambda_{n+1}|\|x\|,$$

which proves (b).

(v) Here let me prove that we found all nonzero eigenvalues. Assume opposite, let $Ap=\lambda p$, $\lambda\neq 0$ and $\lambda\neq\lambda_n$ for all n. We know that p must be orthogonal to all p_k. Using (iv)

$$Ap=\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_k\,\langle p\,,p_k\rangle\,p_k=0,$$

which is a contradiction to $\lambda\neq 0$.

(vi) Here I will show that each eigenvalue has a finite multiplicity and all the eigenvectors corresponding to λ_n were found. Finite multiplicity follows from the fact that $\lambda_n\to 0$. Consider now, say, k eigenvectors $p_1,\ldots,p_k$ that correspond to the same eigenvalue λ. I need to show that if $Ap=\lambda p$ then $p\in\operatorname{span}(p_n)_{n=1}^{k}$. For each n I have

$$Ap_n=\lambda p_n.$$

Therefore, $\operatorname{span}(p_n)\subseteq\{p\in X\colon Ap=\lambda p\}$. To prove the claim I need to show that this inclusion is an equality. Assume contrary, there is $\hat p$ such that $A\hat p=\lambda\hat p$, and $\hat p\notin\operatorname{span}(p_n)$. I can use the Gram-Schmidt orthogonalization to produce $p\in\operatorname{span}\{(p_n),\hat p\}$ and $\langle p\,,p_n\rangle=0$ for any n. Moreover, $Ap=\lambda p$, because p is a linear combination of eigenvectors with the eigenvalue λ. I also have that $\langle p\,,p_n\rangle=0$ for all $n\geq 1$ because A is self adjoint. Now use the same argument as in (v) to reach a contradiction.

(vii) To prepare for the final step I will show that that

$$\ker A=(\operatorname{span}(p_n))^{\perp}.$$

Here $Z^{\perp}$ is the orthogonal complement of $Z \subseteq X$, defined as $Z^{\perp} := \{x \in X \colon \langle x\,, z\rangle = 0,\ z \in Z\}$.

Let $x \in X$ be such that $Ax = 0$. Then for any $n \geq 1$,

$$\langle x\,, p_n\rangle = \frac{1}{\lambda_n}\langle x\,, \lambda_n p_n\rangle = \frac{1}{\lambda_n}\langle Ax\,, p_n\rangle = 0,$$

which means that $x \in (\operatorname{span}(p_n))^{\perp}$. Conversely, if $x \in (\operatorname{span}(p_n))^{\perp}$ then $\langle x\,, p_n\rangle = 0$ for all $n \geq 1$ and thus $Ax = 0$ by (iv).

(ix) Finally, I assume that A is injective, meaning that $\ker A = \{0\}$. By the previous this means that

$$(\operatorname{span}(p_n))^{\perp} = \{0\},$$

that is, the orthogonal complement of $\operatorname{span}(p_n)$ is trivial, meaning that the only vector that is orthogonal to all p_n is zero. This implies that the orthogonal family (p_n) is maximal, and hence, assuming that our Hilbert space is separable, is a basis in X.

This concludes the proof. ■

Remark 5.32. If the range of A is finite dimensional, of dimension N, then I can show that there are exactly N real eigenvalues that satisfy

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_N| > 0,$$

with the corresponding orthogonal eigenvectors $(p_n)_{n=1}^{N}$. Other statements in the proven theorem should be modified to change ∞ for N.

5.9 Sturm–Liouville operators. Symmetric operators

Finally, I can collect all the bits of the information to present a reasonably full and uniform treatment of the following eigenvalue problem. I will study the problem

$$Lu = \lambda u, \tag{5.4}$$

where L is a differential operator

$$Lu = \frac{1}{r(x)}\big(-(p(x)u')' + q(x)u\big). \tag{5.5}$$

I assume that p, p', q, and r are continuous real-valued functions on $[a, b]$, $p(x) > 0$ and $r(x) > 0$ for all $x \in [a, b]$.

Let $\mathcal{H} = \mathcal{L}^2_r$ be a complex Hilbert space, $\mathcal{H} = \{u \colon \int_a^b |u(x)|^2 r(x)\,\mathrm{d}x < \infty\}$ with the inner product

$$\langle u\,, v\rangle = \int_a^b u(x)\overline{v(x)}r(x)\,\mathrm{d}x.$$

The form of operator (5.5) may look too special, but in reality I am dealing with general second order differential operator such that the leading coefficient does not take any zero values on my interval $[a, b]$. Specifically, consider the following exercise.

Exercise 5.26. Consider the eigenvalue problem for general second order differential operator ($p_0(x) \neq 0$ for $x \in [a, b]$)

$$p_0(x)u'' + p_1(x)u' + p_2(x)u = \lambda u.$$

Considering the integrating factor $\exp \int_a^b \frac{p_1(s)}{p_0(s)} \, \mathrm{d}s$ rewrite it in form (5.5). What are p, q, r?

Let D_L be the set of all twice continuously differentiable functions on $[a, b]$, satisfying the boundary conditions

$$B_1 u = \beta_1 u(a) + \gamma_1 u'(a) = 0, \quad B_2 u = \beta_2 u(b) + \gamma_2 u'(b) = 0, \tag{5.6}$$

where $|\beta_1| + |\gamma_1| > 0$ and $|\beta_2| + |\gamma_2| > 0$. Clearly, $D_L \subset \mathcal{H}$, and D_L is the domain of my differential operator (hence the subscript L).

Definition 5.33. *The eigenvalue problem* (5.4), (5.5), (5.6) *is called the regular Sturm–Liouville problem, and operator L defined by* (5.5), (5.6) *is called the Sturm–Liouville operator.*

There were a number of Sturm–Liouville problems in motivational sections in this chapter, if the student do not recognize them, they should go back and find explicit examples.

It is very tempting to use the spectral theorem for compact self-adjoint operators, and conclude that L has a countable number of eigenvalues and eigenvectors that form a basis for $\mathcal{H}$, however, note that L is not even defined on all $\mathcal{H}$, since there are a lot of functions in $\mathcal{H}$ which are not twice continuously differential. For the following exposition, however, it is important to mention that L is *densely defined* on $\mathcal{H}$ because the set D_L is dense in $\mathcal{H}$ (I state this fact without proof; this is not a very restrictive requirement since it can be proved that the set of infinitely differentiable functions with compact support denoted $\mathcal{C}_0^\infty$, is dense in corresponding $\mathcal{L}^2$ space, and hence any set that contains $\mathcal{C}_0^\infty$ as a subset is dense in $\mathcal{L}^2$).

But even this is not all. A differential operator is an example of an *unbounded* operator (think of $A := \frac{\mathrm{d}}{\mathrm{d}x}$ acting on the sequence $(\sin nx)$). Therefore, the Sturm–Liouville operator is unbounded and none of the studied machinery will work for it.

However, Sturm–Liouville operator is *symmetric*.

Definition 5.34. *Linear operator $L \colon D_L \longrightarrow R_L$ is called symmetric if D_L is dense in $\mathcal{H}$, $R_L \subseteq \mathcal{H}$, and*

$$\langle Lu \, , v \rangle = \langle u \, , Lv \rangle$$

for all $u, v \in D_L$.

Carefully note that the definition depends on the domain D_L and also in general includes the boundary conditions.

Lemma 5.35. *The Sturm–Liouville operator is symmetric.*

Proof. Let $u, v \in D_L$. Then we have

$$\begin{aligned} \langle Lu \, , v \rangle - \langle u \, , Lv \rangle &= \int_a^b \big(-(pu')'\overline{v} + qu\overline{v} + (p\overline{v}')'u - qu\overline{v} \big) \, \mathrm{d}x \\ &= \int_a^b \big(-(pu')'\overline{v} + (p\overline{v}')'u \big) \, \mathrm{d}x. \end{aligned}$$

By integrating by parts, I find

$$\langle Lu\,, v\rangle - \langle u\,, Lv\rangle = p(u\overline{v}' - u'\overline{v})|_a^b.$$

From (5.6) I have, since the constants are real, that if $u, v \in D_L$ then $\overline{u}, \overline{v} \in D_L$:

$$\beta_1 u(a) + \gamma_1 u'(a) = 0,$$
$$\beta_1 \overline{v}(a) + \gamma_1 \overline{v}'(a) = 0,$$

and hence the determinant

$$\begin{vmatrix} u(a) & u'(a) \\ \overline{v}(a) & \overline{v}'(a) \end{vmatrix}$$

must vanish. The same is true for

$$\begin{vmatrix} u(b) & u'(b) \\ \overline{v}(b) & \overline{v}'(b) \end{vmatrix}.$$

Therefore,

$$\langle Lu\,, v\rangle - \langle u\,, Lv\rangle = 0.$$

■

Remark 5.36. I actually showed that L is symmetric if and only if $p(u\overline{v}' - u'\overline{v})|_a^b = 0$.

Exercise 5.27. Consider operator $L = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ defined on $\mathcal{C}^{(1)}$ functions that are $2l$-periodic with standard inner product. Show that it is symmetric.

Exercise 5.28. Consider $L = \frac{\mathrm{d}^4}{\mathrm{d}x^4}$ with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$ defined on 4 times continuously differentiable functions with standard inner product. Show that it is symmetric. Also show that it is positive meaning that $\langle Lu\,, u\rangle > 0$ for all $u \neq 0, u \in D_L$.

Exactly as for the self-adjoint operators, it can be proved

Lemma 5.37. *Let L be a symmetric operator on a Hilbert space $\mathcal{H}$. Then if λ is an eigenvalue of L then $\lambda \in \mathbf{R}$. If p_1 and p_2 are two eigenvectors of L corresponding to distinct λ_1 and λ_2 then $\langle p_1\,, p_2\rangle = 0$.*

Exercise 5.29. Prove this lemma.

Everything is ready for the main result of this chapter.

Theorem 5.38. *Let L be a Sturm–Liouville operator* (5.5), (5.6). *Then there exists a sequence of real numbers (μ_n) and an orthonormal basis (ϕ_n) of $\mathcal{H}$ such that each ϕ_n is a $\mathcal{C}^{(2)}$ function, and $L\phi_n = \mu_n\phi_n$. Moreover, each real number μ appears at most twice in the sequence (μ_n) and $|\mu_n| \to \infty$. Furthermore, each function $v \in \mathcal{H}$ can be written in terms of its Fourier series*

$$v = \sum_{n=1}^{\infty} \langle v\,, \phi_n\rangle\, \phi_n.$$

Proof. (i) As my first step I note that without loss of generality I can assume that $\mu = 0$ is not an eigenvalue of my problem $Lu = \mu u$. Indeed, if this to happen (if my boundary value problem would have a zero eigenvalue) then instead of L I would consider the operator $L - \lambda I$ for some real λ. This operator would still be a Sturm–Liouville operator (exercise!) and would have eigenvalues $\mu - \lambda$. Now, since L is symmetric, I cannot have more than countable number of eigenfunctions (they must be orthogonal) and hence a countable number of eigenvalues, hence there is always real λ such that $\mu - \lambda$ is not zero for all eigenvalues μ of my original problem.

(ii) Since $\mu = 0$ is not an eigenvalue of my problem then the problem $Lu = 0$ (including the boundary conditions!) has only trivial solution. Therefore, by Theorem 5.7, there exists Green's function of the problem (I manipulate slightly my L to put it into a more convenient form)

$$-u'' - \frac{p'(x)}{p(x)}u'(x) + \frac{q(x)}{p(x)}u(x) = \mu\frac{r(x)}{p(x)}u(x), \quad B_1u = B_2u = 0.$$

My aim is to find Green's function for

$$-u'' - \frac{p'(x)}{p(x)}u'(x) + \frac{q(x)}{p(x)}u(x) = 0, \quad B_1u = B_2u = 0.$$

Since this is second order differential equation there must exist a fundamental solution set $\{u_1, u_2\}$, which I choose in the way that $B_1u_1 = B_2u_2 = 0$. I can always do this and still have only trivial solution for the whole problem by assuming that $B_1u_2 \neq 0$ and $B_2u_1 \neq 0$. In this case the system

$$\begin{aligned} B_1u &= c_1B_1u_1 + c_2B_1u_2 = 0, \\ B_2u &= c_1B_2u_1 + c_2B_2u_2 = 0, \end{aligned}$$

will still have nonzero determinant and hence $c_1 = c_2 = 0$, that is, only trivial solution.

Due to my assumptions I must have

$$g(x,\xi) = \begin{cases} A(\xi)u_1(x), & a \leq x \leq \xi, \\ B(\xi)u_2(x), & \xi \leq x \leq b. \end{cases}$$

Using the continuity and jump conditions I have the system ($C(\xi) = -B(\xi)$)

$$\begin{aligned} A(\xi)u_1(\xi) + C(\xi)u_2(\xi) &= 0, \\ A(\xi)u_1'(\xi) + C(\xi)u_2'(\xi) &= 1, \end{aligned}$$

whose determinant is the Wronskian $W(\xi)$ of my fundamental solution set and hence nonzero at any point ξ. Solving the system yields

$$A(\xi) = -u_2(\xi)/W(\xi), \quad B(\xi) = -u_1(\xi)/W(\xi).$$

Therefore, by Remark 5.10, my problem is equivalent to the integral equation

$$u(x) = \mu\int_a^b g(x,\xi)\frac{r(\xi)}{p(\xi)}u(\xi)\,\mathrm{d}\xi,$$

where

$$g(x,\xi) = \begin{cases} -\frac{u_1(x)u_2(\xi)}{W(\xi)}, & a \leq x \leq \xi, \\ -\frac{u_1(\xi)u_2(x)}{W(\xi)}, & \xi \leq x \leq b. \end{cases}$$

Finally, by Liouville's formula (see Section 3.8.1), I have that

$$W(\xi) = W(a) \exp\left(-\int_a^\xi \frac{p'(x)}{p(x)}\,\mathrm{d}x\right) = \frac{W(a)p(a)}{p(\xi)},$$

and hence my integral equation becomes

$$u(x) = \mu \int_a^b h(x,\xi) r(\xi) u(\xi)\,\mathrm{d}\xi, \tag{5.7}$$

where

$$p(a)W(a)h(x,\xi) = \begin{cases} -u_1(x)u_2(\xi), & a \leq x \leq \xi, \\ -u_1(\xi)u_2(x), & \xi \leq x \leq b. \end{cases}$$

By introducing new notation $\sqrt{r(x)}h(x,\xi)\sqrt{r(\xi)} = k(x,\xi)$ and $v(x) = \sqrt{r(x)}u(x)$ I have

$$v(x) = \mu \int_a^b k(x,\xi) v(\xi)\,\mathrm{d}\xi.$$

By construction k is real and symmetric, $k(x,\xi) = k(\xi,x)$. Therefore, I can consider (5.7) as an eigenvalue problem

$$Av = \lambda v,$$

where A is the inverse operator to my differential operator L, $A = L^{-1}$, and $\lambda = \mu^{-1}$. By construction A is self-adjoint compact operator.

(*iii*) The obtained above operator $A\colon \mathcal{H} \longrightarrow \mathcal{H}$ is injective. Indeed, if one assumes that the kernel of A is nontrivial, then, using

$$LAv = v$$

on one hand and $L0 = 0$ on the other hand, I reach a contradiction. Therefore, I am allowed to use the result of Theorem 5.31 to guarantee that there is a countable set of eigenvalues λ_n and maximal orthonormal system (ϕ_n). By constructions ϕ_n are the eigenfunctions of L (up to a multiplicative factor) with eigenvalues $\mu_n = 1/\lambda_n$. Since $\lim_{n\to\infty} \lambda_n = 0$ I know that $|\mu_n| \to \infty$. Since second order linear differential equation cannot have more than two linearly independent solutions, then any μ appears no more than twice in the sequence (ϕ_n). Since ϕ_n solve $L\phi_n = \mu_n\phi_n$ then $\phi_n \in \mathcal{C}^{(2)}$. ∎

Remark 5.39. Actually quite more can be proven for the regular Sturm–Liouville problem. In particular, it can be proven additionally that

1. All eigenvalues are simple;
2. There are only finitely many negative eigenvalues, and hence all eigenvalues can be arranged as $\mu_1 < \mu_2 < \ldots < \mu_n < \ldots$, $\mu_n \to \infty$;
3. Each eigenfunction ϕ_n has exactly $n-1$ roots in $[a,b]$;
4. Between two successive zeroes of ϕ_n and also between a and the first zero and the last zero and b there is exactly one zero of ϕ_{n+1}.

Exercise 5.30. Everything above was about the regular Sturm–Liouville problem with unmixed boundary conditions, when one condition corresponds to the left point a of the interval, and the other one to the right point b of the interval. Very similar theorem (up to some minor modifications) can be proved for regular Sturm–Liouville problem with periodic boundary conditions

$$u(a) = u(b), \quad u'(a) = u'(b).$$

Study this case, formulate the correct theorem, and provide all the details of the proof.

Exercise 5.31. The regular Sturm–Liouville problem can be also generalized for differential operators of other orders. Prove that the symmetric operators

1. $L = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$, $u(0) = u(1)$;
2. $L = \frac{\mathrm{d}^4}{\mathrm{d}x^4}$, $u(0) = u'(0) = u(1) = u'(1) = 0$

have a countable set or eigenvalues and eigenfunctions; moreover the eigenfunctions form a basis of $\mathcal{L}^2[0, 1]$.

Corollary 5.40. *The followings orthogonal sets of functions in the respective vector spaces V are complete, that is the corresponding Fourier series of any $\phi \in V$ converge in $\mathcal{L}^2$ to ϕ:*

1. $V = \mathcal{L}^2[0, l]$, $\phi_n(x) = \sin \frac{n\pi x}{l}$, $n = 1, 2, 3, \ldots$;
2. $V = \mathcal{L}^2[0, l]$, $\phi_n(x) = \cos \frac{n\pi x}{l}$, $n = 0, 1, 2, 3, \ldots$;
3. $V = \mathcal{L}^2[-l, l]$, $\phi_n(x) = \exp\left(\frac{\mathrm{i}n\pi x}{l}\right)$, $n \in \mathbf{Z}$;
4. $V = \mathcal{L}^2[-l, l]$, $\phi_n(x) = \cos \frac{n\pi x}{l}$, $n = 0, 1, \ldots, \psi_n(x) = \sin \frac{n\pi x}{l}$, $n = 1, 2, \ldots$

Proof. In the first two cases these are eigenfunctions of the regular Sturm–Liouville problems

$$-u'' + \lambda u = 0, \quad u(0) = u(l)$$

and

$$-u'' + \lambda u = 0, \quad u'(0) = u'(l)$$

correspondingly.

In the third case these are the eigenfunctions of

$$-\mathrm{i}u' + \lambda u = 0, \quad u(-l) = u(l),$$

which is covered in Exercise 5.31.

In the fourth case these are the eigenfunctions of

$$-u'' + \lambda u = 0, \quad u(-l) = u(l),\, u'(-l) = u'(l),$$

which is covered in Exercise 5.30. ■

5.10 Appendix

5.10.1 A little on singular Sturm–Liouville problem